Math 335 Sample Problems

One notebook sized page of notes will be allowed on the test. The test will cover through §7.3

1. Suppose $a_n > 0,\ b_n > 0$ for all $n > 1$. Suppose that $\sum_1^\infty b_n$ converges and that $\frac{a_{n+1}}{a_n} \leq \frac{b_{n+1}}{b_n}$ for $n \geq N$. Prove that $\sum_1^\infty a_n$ converges.

2. Let S be the set of all positive integers whose decimal representation does *not* contain 2. Prove that $\sum_{n \in S} \frac{1}{n}$ converges.

3. Assume $a_n \geq 0$ for all $n \geq 1$. Prove that if $\sum_1^\infty a_n$ converges then $\sum_1^\infty \sqrt{a_n a_{n+1}}$ converges. Give an example of a sequence $a_n \geq 0$ such that $\sum_1^\infty \sqrt{a_n a_{n+1}}$ converges and $\sum_1^\infty a_n$ diverges.

4. Prove that if $\sum_1^\infty a_n$ converges then $\sum_1^\infty \frac{\sqrt{a_n}}{n}$ converges. (Assume $a_n \geq 0$.)

5. Let x_n be a convergent sequence and let $c = \lim_{n\to\infty} x_n$. Let p be a fixed positive integer and let $a_n = x_n - x_{n+p}$. Prove that $\sum a_n$ converges and
$$\sum_1^\infty a_n = x_1 + x_2 + \ldots x_p - pc.$$

6. Suppose $\sum_0^\infty a_n$ converges. Prove that $\sum_0^\infty \frac{a_n}{n+1}$ converges and
$$\int_0^1 \sum_0^\infty a_n x^n dx = \sum_0^\infty \frac{a_n}{n+1}.$$

7. (a) Suppose f_n converges uniformly on S. Prove that $|f_n|$ converges uniformly on S.

(b) Suppose f_n is Riemann integrable on $I \subset \mathbb{R}$. Assume that f_n converges uniformly on I to f. Prove that

$$\lim_{n\to\infty} \int_I f_n^2 = \int_I f^2.$$

(c) Suppose f_n converges uniformly on S. Does f_n^2 converge uniformly on S? Give a proof or counterexample.

8. (a) Suppose $\sum_1^\infty a_n$ converges. Fix $p \in \mathbb{Z}^+$. Prove that $\lim_{n\to\infty}(a_n + a_{n+1} + \ldots a_{n+p}) = 0$.
   (b) Suppose $\lim_{n\to\infty}(a_n + a_{n+1} + \ldots a_{n+p}) = 0$ for every p. Does $\sum_1^\infty a_n$ converge?

9. Let $a_n > 0$ and suppose $a_n \geq a_{n+1}$. Prove that $\displaystyle\sum_1^\infty a_n$ converges if and only if $\displaystyle\sum_1^\infty a_{3n}$ converges.

10. Let $a_n > 0$ and let

$$L_n = \left[\log(\frac{1}{a_n})\right] / (\log n).$$

Assume $L = \lim_{n\to\infty} L_n$ exits.

   (a) If $L > 1$ prove that $\sum_n a_n$ converges.
   (b) If $L < 1$ prove that $\sum_n a_n$ diverges.

11. Suppose f is continuous on $[0, a]$. Let f_n be defined inductively by

$$f_0(x) = f(x), f_{n+1}(x) = \int_0^x f_n(t)dt.$$

Prove that $f_n \to 0$ uniformly on $[0, a]$.

12. Prove that

$$\frac{1}{n!} > \sum_{j=n+1}^{\infty} \frac{1}{j!},$$

for $n \geq 1$.

13. Suppose that $a_n \geq 0$ and $\displaystyle\sum_{n=0}^\infty a_n$ diverges; and suppose that $\displaystyle\sum_{n=0}^\infty a_n x^n$ converges for $|x| < 1$. Prove

$$\lim_{x\to 1^-} \sum_{n=0}^\infty a_n x^n = +\infty.$$

14. Suppose f_n is a sequence of continuous functions that converges uniformly on a set W. Let p_n be a sequence of points in W that converges to a point $p \in W$. Prove that $\lim_{n\to\infty} f_n(p_n) = f(p)$.

15. Let be a sequence of continuous functions in $I = [a, b]$ and suppose $f_n(x) \geq f_{n+1}(x) \geq 0$ for all $x \in I$. Suppose $\displaystyle\lim_{n\to\infty} f_n(x) = 0$ for all $x \in I$ (point-wise convergence to 0). Is the convergence uniform? Give a proof or a counterexample.

16. Prove that $\displaystyle\sum_{n=0}^{\infty} \frac{x}{(1+|x|)^n}$ converges for all x, but the convergence is not uniform.

17. Assume $p \geq 1,\ q \geq 1$. Prove that

$$\int_0^1 \frac{t^{p-1}}{1+t^q} dt = \frac{1}{p} - \frac{1}{p+q} + \frac{1}{p+2q} \cdots .$$

Give careful justification of any manipulations.

18. Suppose $a_n > b_n > 0$, $a_n > a_{n+1}$ and $\lim_{n\to\infty} a_n = 0$. Does $\displaystyle\sum_1^{\infty} (-1)^n b_n$ converge? Give a proof or a counterexample.

19. Prove that $\displaystyle\sum_{n=1}^{\infty} \frac{\cos nx}{n}$ converges uniformly for $x \in [a,b], 0 < a < b < 2\pi$, but does not converge absolutely for any x.

20. Prove that $\displaystyle\sum_1^{\infty} (-1)^n \frac{\sin nx}{n}$ converges uniformly on $\{|x| < 1\}$ to a continuous function.

21. Let f_n be a sequence of functions defined on the open interval (a,b). Suppose $\lim_{x\to a^+} f_n(x) = a_n$ for all n. Suppose $\displaystyle\sum_1^{\infty} f_n$ converges uniformly on (a,b) to a function f. Prove that $\displaystyle\sum_1^{\infty} a_n$ converges and $\displaystyle\lim_{x\to a^+} f(x) = \sum_1^{\infty} a_n$. Do not assume f_n is continuous on (a,b).

22. Suppose the series $\sum_1^{\infty} a_n$ converges. Prove that $\sum_1^{\infty} \frac{a_n}{n^x}$ converges for $x \geq 0$. Let $f(x) = \sum_1^{\infty} \frac{a_n}{n^x}$. Prove that $\lim_{x\to 0^+} f(x) = \sum_1^{\infty} a_n$.

23. Let $p_j(t) = e^{-t}\dfrac{t^j}{j!}$.

    (a) Suppose $\sum_0^{\infty} a_n$ converges. Let $s_n = \sum_0^n a_j$. Prove that

    $$\lim_{t\to\infty} \sum_0^{\infty} s_j p_j(t) = \sum_0^{\infty} a_n.$$

    (b) Compute this limit in the case that $a_n = x^n$ for those x for which the limit exists (even in the case that $\sum x^n$ does not converge). This limit is called the Borel regularized value. What does this give for the *Borel regularized value* of $1 - 2 + 4 - 8 + 16 \pm \ldots$?

24. You will need to know the definitions of the following terms and statements of the following theorems.

(a) Convergence and divergence of a series

(b) Comparison test

(c) Integral test

(d) Cauchy condensation test

(e) Root test and ratio test

(f) Abel's theorem

(g) Uniform convergence of a sequence or series of functions

(h) Weierstrass M-test

(i) Continuity of a uniform limit of continuous functions

(j) Integration and differentiation of a sequence or series

(k) Power series

(l) Radius of convergence of a power series

(m) Integration and differentiation of a power series

25. There may be homework problems or example problems from the text on the midterm.